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ORIGINAL PAPER

A note on Li–Yorke chaos in a coupled lattice system related with Belusov–Zhabotinskii reaction

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Abstract This paper is concerned with the following system which comes from a lattice dynamical system stated by Kaneko in (Phys Rev Lett 65:1391–1394, 1990) and is related to the Belusov–Zhabotinskii reaction:

$$x_n^{m+1} = (1-\varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m) \right],$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), ε is coupling constant, and $f(x)$ is the unimodal map on I (i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ and $f(c) = 1$). It is proved that for coupling constant $\varepsilon = 1$, this CML (Coupled Map Lattice) system is chaotic in the sense of Li–Yorke for each unimodal selfmap on the interval $I = [0, 1]$.

Keywords Coupled map lattice · Distributional (p, q)-chaos · Principal measure · Devaney's chaos · Chaos in the sense of Li–Yorke · Tent map

1 Introduction

By a topological dynamical system (t.d.s. for short) (X, f) we mean a compact metric space X together with a continuous map $f : X \to X$. Since Li and Yorke [1] introduced

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the term of chaos in 1975, the dynamical properties in many topological dynamical systems were highly discussed and studied in the literature(see [2,3]) because they are good important examples of problems coming from topological dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

Coming from physical and chemical engineering applications, e.g., a digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, The so called Lattice Dynamical Systems or 1d Spatiotemporal Discrete Systems generalize t.d.s.'s, and they have recently appeared as an important subject for investigation. From [4] one can easily see the importance of these type of systems.

To understand when one of this type of systems has a complicated dynamics or not by the consideration and observation of one topological dynamical property is an open and interesting problem (see [5]). In [5], by using the notion of chaos the authors studied and characterized the dynamical complexity of a coupled lattice system stated by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction. And they showed that this CML (Coupled Map Lattice) system is chaotic in the sense of Li–Yorke and in the sense of Devaney for zero coupling constant. Also, some problems on the dynamics of the CML system with non-zero coupling constant are stated. Recently, it was showed that this system with non-zero coupling constant is chaotic in the sense of Li–Yorke and has positive topological entropy (see [7]).

In [8], the notion of distributional chaos introduced by Schweizer and Smítal is very interesting and important. On the one hand this concept is equivalent to positive topological entropy and some other concepts of chaos when restricted to the compact interval case [8] or hyperbolic symbolic spaces [9], but on the other hand this equivalence does not transfer to higher dimensions, e.g., positive topological entropy does not imply distributional chaos in the case of triangular maps of the unit square [10] (the same happens when the dimension is zero [11]). In [12] the authors presented a distributional chaotic minimal system. More recently, it was shown that the following coupled lattice system with non-zero coupling constant is distributionally (p, q)-chaotic for any pair $0 \leq p \leq q \leq 1$ (see [13]):

$$x_n^{m+1} = (1-\varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m) \right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), ε is coupling constant and $f(x)$ is the unimodal map on I (i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ and $f(c) = 1$.) Inspired by the above results, we will further investigate the dynamical properties of this lattice dynamical systems with coupling constant $\varepsilon = 1$. In particular, we prove that for coupling constant $\varepsilon = 1$, this CML (Coupled Map Lattice) system is chaotic in the sense of Li–Yorke for any unimodal selfmap on the interval $I = [0, 1]$.

2 Preliminaries

Firstly we complete some notations and recall some concepts. Throughout this paper, $I = [0, 1]$.

A pair of points $x, y \in X$ is called a Li–Yorke pair of system (X, f) if the following two conditions hold:

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$, and
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

A subset $S \subset X$ is called a LY-scrambled set for f (Li–Yorke set) if the set S has at least two points and every pair of distinct points in S is a Li–Yorke pair. A system (X, f) or a map $f : X \to X$ is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set.

Let (X, f) be a t.d.s.. For any pair of points $x, y \in X$ and for any $n \in \mathbb{N}$, the distributional function $F^n_{xy} : \mathbb{R}^+ \to [0, 1]$ is defined by

$$F^n_{xy}(t) = \frac{1}{n}\sharp\left\{i \in \mathbb{N} : d(f^i(x), f^i(y)) < t, 1 \le i \le n\right\},$$

where $\mathbb{R}^+ = [0, +\infty)$ and $\sharp A$ denotes the cardinality of A. Let

$$F_{xy}(t, f) = \liminf_{n\to\infty} F^n_{xy}(t)$$

and

$$F^*_{xy}(t, f) = \limsup_{n\to\infty} F^n_{xy}(t).$$

Given $0 \le p \le q \le 1$, a t.d.s. (X, f) is distributionally (p, q)-chaotic if there is an uncountable subset $S \subset X$ and $\varepsilon > 0$ satisfying that $F_{xy}(t, f) = p$ and $F^*_{xy}(t, f) = q$ for any pair of distinct points $x, y \in S$ and any $t \in (0, \varepsilon)$. In particular, (X, f) is distributionally chaotic if it is distributionally $(0, 1)$-chaotic (see [13,14]).

The state space of Lattice Dynamical System is the set

$$\mathcal{X} = \left\{x : x = \{x_i\}, x_i \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|x_i\| < \infty\right\}.$$

where $a \ge 1$ is the dimension of the range space of the map of state x_i, $b \ge 1$ is the dimension of the lattice and the l^2 norm

$$\|x\|_2 = \left(\sum_{i\in\mathbb{Z}^b} |x_i|^2\right)^{\frac{1}{2}}$$

is usually taken ($|x_i|$ is the length of the vector x_i) (see [5]).

We will explore the following Coupled Map Lattice system stated by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction (for this point one can see [15], and for experimental study of chemical turbulence by this method, one can see [16–18]):

$$x^{m+1}_n = (1-\varepsilon) f(x^m_n) + \frac{1}{2}\varepsilon[f(x^m_{n-1}) + f(x^m_{n+1})], \tag{2}$$

where m is discrete time index, n is lattice side index with system size L, ε is coupling constant and $f(x)$ is the unimodal selfmap on I.

In general, one of the following periodic boundary conditions of the systems (1) and (2) is assumed:

1) $x_n^m = x_{n+L}^m$,
2) $x_n^m = x_n^{m+L}$,
3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

3 Main results

The system (1) were investigated by many authors. The first paper with analytic results is [19], where the authors proved that the system (1) is chaotic in the sense of Li–Yorke.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L)) = \left(\sum_{i=1}^{L}(x_i - y_i)^2\right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L) \in I^L$.

Now we define a map $F : (I^L, d) \to (I^L, d)$ by $F(x_1, x_2, \ldots, x_L) = (y_1, y_2, \ldots, y_L)$ where $y_i = (1-\varepsilon)f(x_i) + \frac{\varepsilon}{2}(f(x_{i-1}) - f(x_{i+1}))$. It is noted that the system (1) is equivalent to the system (I^L, F) (see [13]).

Let d' be the product metric on the product space I^L defined by

$$d'((x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L)) = \max\{|x_i - y_i| : i \in \{1, 2, \ldots, L\}\}$$

for any $(x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L) \in I^L$. Obviously, the product metric d' is equivalent to the product metric d.

Inspired by Theorem 3 in [7] and its proof we have the following result.

Theorem 3.1 *The system (1) is chaotic in the sense of Li–Yorke for couplings constant $\varepsilon = 1$ and any unimodal map on the unite closed interval I.*

Proof By Theorem 2 in [7] there is an uncountable scrambled set $S \subset I$ of f. Write

$$D = \left\{\overrightarrow{x} = (x, x, \ldots, x) \in I^L : x \in S\right\}.$$

Then, for any pair of points $\overrightarrow{x} = (x, x, \ldots, x), \overrightarrow{y} = (y, y, \ldots, y) \in D$ with $\overrightarrow{x} \neq \overrightarrow{y}$ and any $n \in \mathbb{N}$ we obtain that

$$F^n(\overrightarrow{x}) = \overrightarrow{f^n(x)} = (f^n(x), f^n(x), \ldots, f^n(x))$$

and

$$F^n(\overrightarrow{y}) = \overrightarrow{f^n(y)} = (f^n(y), f^n(y), \ldots, f^n(y)).$$

By the definition

$$d'(F(\overrightarrow{x}), F(\overrightarrow{y})) = |f(x) - f(y)|$$

for any $x, y \in I$. So, we have

$$\limsup_{n\to\infty} d'(F^n(\overrightarrow{x}), F^n(\overrightarrow{y})) = \limsup_{n\to\infty} |(f^n(x) - f^n(y)| > 0$$

and

$$\liminf_{n\to\infty} d'(F^n(\overrightarrow{x}), F^n(\overrightarrow{y})) = \liminf_{n\to\infty} |(f^n(x) - f^n(y)| = 0$$

for any $x, y \in S$. Therefore, By the definition $D \subset I^L$ is an uncountable scrambled set of F. Thus, the proof is finished. □

Remark 3.1 It is known from [7] that the system (1) is chaotic in the sense of Li–Yorke for any $\varepsilon \in (0, 1)$ and any unimodal selfmap of the interval I. However, by Theorem 3.1, the system (1) is also chaotic in the sense of Li–Yorke for $\varepsilon = 1$ and any unimodal selfmap of the space [0, 1].

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